Diamagnetic expansions for perfect quantum gases II: uniform bounds.

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Abstract: Consider a charged, perfect quantum gas, in the effective mass approximation, and in the grand-canonical ensemble. We prove in this paper that the generalized magnetic susceptibilities admit the thermodynamic limit for all admissible fugacities, uniformly on compacts included in the analyticity domain of the grand-canonical pressure.

The problem and the proof strategy were outlined in [3]. In [4] we proved in detail the pointwise thermodynamic limit near $z = 0$. The present paper is the last one of this series, and contains the proof of the uniform bounds on compacts needed in order to apply Vitali's Convergence Theorem.

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1 Introduction and results.

The magnetic properties of a charged perfect quantum gas in the independent electron approximation and confined to a box Λ have been extensively studied in the literature. One of the central problems has been to establish the thermodynamic limit for the magnetization and magnetic susceptibility, see e.g. [1], [2], [3], [4] ,[5], [6], [9] and references therein.

Briefly, the technical question is whether the thermodynamic limit ($\Lambda \to \mathbb{R}^3$) commutes with the derivatives of the grand-canonical pressure with respect to the external constant magnetic field B.

A general way of proving this thermodynamic limit has been already announced in [3] and [6], and the main ingredient consisted in applying the magnetic perturbation theory to a certain Gibbs semigroup. The strategy of the proof, which works not only for the first two derivatives, but also for derivatives of all orders, were outlined in [3]. In [4] we proved in detail the pointwise thermodynamic limit near $z = 0$. This paper is the last one of the series, it contains the complete proof of the uniform bounds on compacts needed in order to apply the Vitali Convergence Theorem (see [10]).

Now let us formulate the mathematical problem. The box which contains the quantum gas will be the cube $\Lambda \subset \mathbb{R}^3$ of side length $L > 0$ centered at 0. The constant magnetic field is $\mathbf{B} = (0, 0, B)$, with $B \geq 0$, oriented parallel to the third component of the canonical basis in $\mathbb{R}^3$.

We associate to $\mathbf{B}$ the magnetic vector potential $B\mathbf{a}(\mathbf{x}) = \frac{B}{2}(-x_2, x_1, 0)$ and the cyclotronic frequency $\omega = \frac{e}{c}B$. In the rest of the paper, ω will be a real parameter. The one particle Hamiltonian we consider is the self-adjoint operator densely defined in $L^2(\Lambda)$:

$$H_L(\omega) := \frac{1}{2}(-i\nabla - \omega\mathbf{a})^2, \tag{1.1}$$

corresponding to Dirichlet boundary conditions.

One denotes by $B_1(L^2(\Lambda))$ the Banach space of trace class operators. At $\omega \geq 0$ fixed, the magnetic Schrödinger operator $H_L(\omega)$ generates a Gibbs semigroup $\{W_L(\beta, \omega)\}_{\beta \geq 0}$ where:

$$W_L(\beta, \omega) := e^{-\beta H_L(\omega)}, \quad \|W_L(\beta, \omega)\|_{B_1} \leq \frac{L^3}{(2\pi\beta)^{\frac{3}{2}}}, \quad \beta > 0. \tag{1.2}$$

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Let us now introduce the grand-canonical formalism. Let $\beta = 1/(kT) > 0$ be the inverse temperature, $\mu \in \mathbb{R}$ the chemical potential and $z = e^{\beta\mu}$ the fugacity. Let K be a compact included in the domain $D_+ := \mathbb{C}\setminus[e^{\frac{\beta\omega}{2}}, \infty[$ for the Bose statistics and $D_- := \mathbb{C}\setminus]-\infty, -e^{\frac{\beta\omega}{2}}]$ for the Fermi case.

Fix $\omega_0 > 0$ and a compact real interval Ω containing ω_0. For a fixed $\beta > 0$, one can find a simple, positively oriented, closed contour $\mathcal{C}_K \subset D_\pm$ whose interior does not contain 1 in the Bose case or -1 in the Fermi case, and such that

$$\sup_{L>1}\sup_{\omega\in\Omega}\sup_{\xi\in\mathcal{C}_K}\sup_{z\in K}\left\|[\xi - zW_L(\beta,\omega)]^{-1}\right\| = M < \infty. \tag{1.3}$$

Details may be found in [5] for the Bose case, but the main idea is that the spectrum of $W_L(\beta,\omega)$ is always contained in the interval $[0, e^{-\beta\omega/2}]$ and one can apply the spectral theorem.

We then can express the grand canonical pressure at ω_0 as follows (see e.g. [5] for the Bose case):

$$P_L(\beta, z, \omega_0) = \frac{-\epsilon}{2i\pi\beta L^3}\int_{\mathcal{C}_K} d\xi\, \frac{\ln(1-\epsilon\xi)}{\xi}\mathrm{Tr}\left[(\xi - zW_L(\beta,\omega_0))^{-1} zW_L(\beta,\omega_0)\right]. \tag{1.4}$$

where $\epsilon = 1$ for the Bose gas, and $\epsilon = -1$ for the Fermi gas.

For $\omega \in \mathbb{R}$, $\xi \in \mathcal{C}_K$ and $z \in K$, introduce the operator:

$$g_L(\beta, z, \xi, \omega) := [\xi - zW_L(\beta,\omega)]^{-1} zW_L(\beta,\omega). \tag{1.5}$$

This is a trace class operator which obeys (use (1.3) and (1.2)):

$$\|g_L(\beta,\omega,\xi,\omega)\|_{B_1} \leq (\sup_{z\in K}|z|)\,\frac{L^3 M}{(2\pi\beta)^{\frac{3}{2}}}. \tag{1.6}$$

uniformly in $\omega \in \Omega$. Because $\omega \to W_L(\beta,\omega)$ is a B_1-entire operator valued function in ω (this result was first obtained in [1] and then refined in [4]), then using (1.3) one easily shows that the map

$$]0,\infty[\ni \omega \mapsto \mathrm{Tr}\ g_L(\beta, z, \xi, \omega) \in \mathbb{C}$$

is smooth, with derivatives which are uniformly bounded in ξ and z. Thus for every $N \geq 1$ and $z \in K$ we can define the generalized susceptibilities at ω_0 by

$$\begin{aligned}\chi_L^N(\beta, z, \omega_0) &:= \frac{\partial^N P_L}{\partial\omega^N}(\beta, z, \omega_0) \\ &= \frac{-\epsilon}{2i\pi\beta L^3}\int_{\mathcal{C}_K} d\xi\,\frac{\ln(1-\epsilon\xi)}{\xi}\ \frac{\partial^N \mathrm{Tr}\ g_L}{\partial\omega^N}(\beta, z, \xi, \omega_0).\end{aligned} \tag{1.7}$$

From this discussion one can see that the pressure as well $\chi_L^N(\beta,\cdot,\omega_0)$, $N \geq 1$ are analytic functions on D_+ (or D_-).

Now let us describe the case when $L = \infty$. The thermodynamic limit of the pressure exists and is uniform on compacts like K. If $\omega > 0$, its actual value is (see [2]):

$$P_\infty(\beta, z, \omega_0) := \omega_0\frac{1}{(2\pi\beta)^{3/2}}\sum_{k=0}^{\infty} f_{3/2}^{\epsilon}\left(ze^{-(k+1/2)\omega_0\beta}\right), \tag{1.8}$$

where $f_\sigma^\epsilon(\zeta)$ are the usual Bose (or Fermi) functions for $\epsilon = 1$ (or $\epsilon = -1$):

$$f_\sigma^\epsilon(\zeta) := \frac{\zeta}{\Gamma(\sigma)}\int_0^\infty dt\,\frac{t^{\sigma-1}e^{-t}}{1-\zeta\epsilon e^{-t}}, \tag{1.9}$$

analytic in $\mathbb{C}\setminus[1,\infty[$ (or $\mathbb{C}\setminus]-\infty,-1]$) if $\epsilon = 1$ (or $\epsilon = -1$). If $|\zeta| < 1$, they are given by the following expansion:

$$f_\sigma^\epsilon(\zeta) = \sum_{n=1}^{\infty}\frac{\epsilon^{n-1}\zeta^n}{n^\sigma}\ .$$

Now it is rather easy to verify that for any $N \geq 0$, the multiple derivative $\partial_\omega^N P_\infty(\beta, \cdot, \omega_0)$ exists and defines an analytic function on D_+ (or D_-). The main result of [3] established the pointwise convergence,

$$\lim_{L\to\infty} \partial_\omega^N P_L(\beta, \cdot, \omega_0) = \partial_\omega^N P_\infty(\beta, \cdot, \omega_0) := \chi_\infty^N(\beta, \cdot, \omega_0); \quad |z| < 1.$$

Remember that we want to apply the Vitali Convergence Theorem (see [10] or [3]). Therefore, in order to conclude that $\chi_L^N(\beta, z, \omega_0)$ converges uniformly to $\chi_\infty^N(\beta, z, \omega_0)$ for all $z \in K$, the only remaining point is to get the uniform boundedness w.r.t. L. More precisely, we will prove:

Theorem 1.1. *For all $N \geq 1$, for all $\beta > 0$ and for all $\omega > 0$,*

$$\sup_{L>1} \sup_{z\in K} \left|\chi_L^N(\beta, z, \omega)\right| \leq \text{const}(\beta, K, \omega, N). \tag{1.10}$$

Then putting this together with the pointwise convergence result near $z = 0$ of [4], the final conclusion would be:

$$\lim_{L\to\infty} \sup_{z\in K} \left|\chi_L^N(\beta, z, \omega_0) - \chi_\infty^N(\beta, z, \omega_0)\right| = 0. \tag{1.11}$$

Remark 1.2. *Having uniform convergence* (1.11) *with respect to z allows us to prove existence of the thermodynamic limit for canonical susceptibilities (see [3, 6])*

This paper is devoted to the proof of Theorem 1.1. Note that the theorem is an immediate consequence of the following estimate:

$$\sup_{\xi\in\mathcal{C}_K} \sup_{z\in K} \left|\frac{\partial^N \text{Tr } g_L}{\partial\omega^N}(\beta, \xi, z, \omega_0)\right| \leq L^3 \,\text{const}(\beta, K, N, \omega_0), \tag{1.12}$$

which would imply via (1.7) that the generalized susceptibilities are uniformly bounded in L.

1.1 Strategy of the proof

From now on, we omit the parameters ξ and z in the definition of g_L in order to simplify notation. Fix $\beta > 0$ and $\omega_0 \geq 0$. Let $\Omega \subset \mathbb{R}$ be a compact interval containing ω_0. If $\omega \in \Omega$, we denote by $\delta\omega := \omega - \omega_0$. The main idea of the proof is to derive an equality of the following type:

$$\text{Tr } g_L(\beta, \omega) = \text{Tr } g_L(\beta, \omega_0) + \sum_{j=1}^{N} (\delta\omega)^j a_j(\beta, \omega_0) + (\delta\omega)^{N+1}\mathcal{R}_L(\beta, \omega, N), \tag{1.13}$$

where the coefficients $a_j(\beta, \omega_0)$ grow at most like L^3 uniformly in ξ and z, while the remainder $\mathcal{R}_L(\beta, \cdot, N)$ is a smooth function near ω_0. Then since we know that Tr $g_L(\beta, \cdot)$ is smooth, we must have

$$\frac{\partial^N \text{Tr } g_L}{\partial\omega^N}(\beta, \omega_0) = N! a_N(\beta, \omega_0),$$

and this would finish the proof. In order to achieve this program, we will have to do two things.

First step: with the help of magnetic perturbation theory we will find a regularized expansion in $\delta\omega$ for g_L of the form

$$g_L(\beta, \omega) = \sum_{n=0}^{N} (\delta\omega)^n g_{L,n}(\beta, \omega) + R_{L,N}(\beta, \omega, N), \tag{1.14}$$

which holds in the sense of trace class operators, and the remainder has the property that $\frac{1}{(\delta\omega)^{N+1}} R_{L,N}(\beta, \omega)$ is smooth near ω_0 in the trace class topology. The operator-coefficients $g_{L,n}(\beta, \omega)$ *will still depend on* ω, but in a more convenient way. That is, they are sums, products, or integrals of products of *regularized operators*, see (2.2). This result is precisely stated in Theorem 3.5.

Second step: show that for each $0 \le n \le N$ we can write

$$\mathrm{Tr}\; g_{L,n}(\beta,\omega) = \sum_{j=0}^{N} (\delta\omega)^j s_{L,j,n}(\beta,\omega_0) + (\delta\omega)^{N+1}\mathcal{R}_{L,N}(\beta,\omega,N), \tag{1.15}$$

where the remainder $\mathcal{R}_{L,n}(\beta,\cdot)$ is smooth near ω_0. Now the coefficients $s_{L,j,n}(\beta,\omega_0)$ are finally independent of ω, and grow at most like L^3. This is done in the last section.

Finally, if we combine (1.15) with (1.14), we immediately obtain (1.13).

Now let us discuss why a more direct approach only based on trace norm estimates cannot work. Recall that the map $\omega \to W_L(\beta,\omega) \in B_1$ is real analytic, hence $\frac{\partial^N W_L}{\partial\omega^N}$ is well defined in $B_1(L^2(\Lambda))$, and we have the estimate (see [1] and [4]) :

$$\left\| \frac{1}{N!}\, \frac{\partial^N W_L}{\partial\omega^N}(\beta,\omega_0)\right\|_{B_1} \le\; c_N\, \frac{L^{3+N}\,(1+\beta)^{sN}}{\beta^{\frac{3}{2}}\left[\frac{N-1}{4}\right]!}, \tag{1.16}$$

where c_N is a positive constant which depends on N, ω_0 and s. Now if we use the Leibniz rule of differentiation for the product which defines the operator $g_L(\beta,\omega)$ (see (1.5)), and estimate traces by trace norms we obtain:

$$\left|\frac{\partial^N \mathrm{Tr}\; g_L}{\partial\omega^N}(\beta,\omega_0)\right| \le \left\|\frac{\partial^N g_L}{\partial\omega^N}(\beta,\omega_0)\right\|_{B_1} \le L^{3+N}\mathrm{const}\,(\beta,K,\omega_0,N). \tag{1.17}$$

Now this is definitely not good enough, and we have to find a more convenient expansion, as described in (1.15). This will be done in the next sections.

2 Regularized expansion of W_L

It has been shown in [3] and [4] that by using gauge invariance one can control the linear growth of the magnetic vector potential $\mathbf{a}$. The price one pays is the introduction of an antisymmetric phase factor, which disappears though when one takes the trace. Let us now show how this works for the operatorW_L.

Fix $\omega_0 \ge 0$ and $\beta > 0$. Let $\omega \in \mathbb{C}$ and $\delta\omega$ as above. Let us define the magnetic phase:

$$\phi(\mathbf{x},\mathbf{x}') := \mathbf{x}\cdot\mathbf{a}(\mathbf{x}') = \frac{1}{2}(x_2x_1' - x_1x_2') = -\phi(\mathbf{x}',\mathbf{x}); \quad (\mathbf{x},\mathbf{x}') \in \Lambda^2. \tag{2.1}$$

If $T(\omega_0)$ is a bounded operator with an integral kernel $t(\cdot,\cdot,\omega_0)$, then the notation $\widetilde{T}(\omega)$ will refer to the regularized operator associated to $T(\omega_0)$ which has the kernel:

$$\widetilde{t}(\mathbf{x},\mathbf{x}',\omega) := e^{i\delta\omega\phi(\mathbf{x},\mathbf{x}')}t(\mathbf{x},\mathbf{x}',\omega_0); \quad (\mathbf{x},\mathbf{x}') \in \Lambda^2. \tag{2.2}$$

We will very often use the Schur-Holmgren criterion of boundedness for integral operators (see [8]), which states that if T has an integral kernel $t(\mathbf{x},\mathbf{x}')$, then:

$$||T|| \le \left\{\sup_{\mathbf{x}'\in\Lambda}\int_\Lambda |t(\mathbf{x},\mathbf{x}')|d\mathbf{x}\;\cdot \sup_{\mathbf{x}\in\Lambda}\int_\Lambda |t(\mathbf{x},\mathbf{x}')|d\mathbf{x}'\;\right\}^{1/2}. \tag{2.3}$$

We denote by $G_L(\cdot,\cdot,\beta,\omega)$ the kernel of $W_L(\beta,\omega)$. We define two other bounded operators $R_{1,L}$ and $R_{2,L}$ by their kernels,

$$\begin{aligned} R_{1,L}(\mathbf{x},\mathbf{x}',\beta) &:= \mathbf{a}(\mathbf{x}-\mathbf{x}')\cdot[i\nabla_{\mathbf{x}} + \omega_0\mathbf{a}\,(\mathbf{x})]\,G_L(\mathbf{x},\mathbf{x}',\beta,\omega_0),\\ R_{2,L}(\mathbf{x},\mathbf{x}',\beta) &:= \frac{\mathbf{a}^2(\mathbf{x}-\mathbf{x}')}{2}\,G_L(\mathbf{x},\mathbf{x}',\beta,\omega_0); \quad (\mathbf{x},\mathbf{x}') \in \Lambda^2. \end{aligned} \tag{2.4}$$

Then consider the corresponding regularized operators $\widetilde{W}_L,\ \widetilde{R}_{1,L},\ \widetilde{R}_{2,L}$. Let us state here two important estimates, the first one is just the diamagnetic inequality, while the second one was obtained in [5],

$$|G_L(\mathbf{x},\mathbf{x}',\beta,\omega_0)| \leq G_\infty(\mathbf{x},\mathbf{x}',\beta,0) = \frac{1}{(2\pi\beta)^{3/2}} \exp\left(-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\beta}\right), \tag{2.5}$$

$$|[i\nabla_{\mathbf{x}} + \omega_0\mathbf{a}(\mathbf{x})]\, G_L(\mathbf{x},\mathbf{x}',\beta,\omega_0)| \leq C(1+\omega_0)^3 \frac{(1+\beta)^5}{\sqrt{\beta}}\, G_\infty(\mathbf{x},\mathbf{x}',8\beta,0), \tag{2.6}$$

on Λ^2, where $C>0$ is a numerical constant. A straightforward application of the Schur-Holmgren criterion gives us the following operator norm estimates

$$\|\widetilde{W}_L(\beta,\omega)\| \leq 1, \quad \|\widetilde{R}_{i,L}(\beta,\omega)\| \leq C_0\,(1+\omega_0)^3(1+\beta)^5, \tag{2.7}$$

where $i=1,2$ and $C_0>0$ is a numerical constant.

For $i_1,...,i_n \in \{1,2\}$, define

$$D_n(\beta) = \{(\tau_1,...,\tau_n)\in\mathbb{R}^n \ :\ 0<\tau_n<...<\tau_1<\beta\}. \tag{2.8}$$

Introduce the operator norm convergent Bochner integrals:

$$\begin{aligned} I_{n,L}(i_1,...,i_n)(\beta,\omega) &:= \int_{D_n(\beta)} d\tau\, \widetilde{W}_L(\beta-\tau_1,\omega)\widetilde{R}_{i_1,L}(\tau_1-\tau_2,\omega) \\ &\cdot \widetilde{R}_{i_2,L}(\tau_2-\tau_3,\omega)...\widetilde{R}_{i_{n-1},L}(\tau_{n-1}-\tau_n,\omega)\widetilde{R}_{i_n,L}(\tau_n,\omega), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} J_{n,L}(\beta,\omega) &:= \int_{D_n(\beta)} d\tau\, W_L(\beta-\tau_1,\omega)\widetilde{R}_L(\tau_1-\tau_2,\omega) \\ &\cdot \widetilde{R}_L(\tau_2-\tau_3,\omega)...\widetilde{R}_L(\tau_{n-1}-\tau_n,\omega)\widetilde{R}_L(\tau_n,\omega). \end{aligned} \tag{2.10}$$

Here we used the notation:

$$\widetilde{R}_L(\beta,\omega) = (\delta\omega)\widetilde{R}_{1,L}(\beta,\omega) + (\delta\omega)^2\widetilde{R}_{2,L}(\beta,\omega). \tag{2.11}$$

By (2.7) the operators $I_{n,L}(i_1,...,i_n)$ and $J_{n,L}$ belong to $B(L^2(\Lambda))$. We will show below that in fact their belong to $B_1(L^2(\Lambda))$ and their trace norm is of order L^3. Denote by $\chi_n^j(i_1,...,i_n)$ the characteristic function of the set

$$\{(i_1,...,i_n)\in\{1,2\}^n \ :\ \sum_{k=1}^n i_k = j\}.$$

Proposition 2.1. *Fix $\omega_0>0$, and $N\geq 1$. Set $\delta\omega=\omega-\omega_0, \omega\in\mathbb{C}$. Then we have the following identity in $B_1(L^2(\Lambda))$:*

$$W_L(\beta,\omega) = \widetilde{W}_L(\beta,\omega) + \sum_{n=1}^N (\delta\omega)^n\, W_{L,n}(\beta,\omega) + R_{L,N}^{(1)}(\beta,\omega), \tag{2.12}$$

where $\omega \to \frac{1}{(\delta\omega)^{N+1}} R_{L,N}^{(1)}(\beta,\omega)$ is a smooth B_1 operator valued function in ω.

The coefficients of the above expansion are given by:

$$W_{L,n}(\beta,\omega) := \sum_{k=1}^n (-1)^k \sum_{i_j\in\{1,2\}} \chi_k^n(i_1,...,i_k) I_{k,L}(i_1,...,i_k)(\beta,\omega), \tag{2.13}$$

and the reminder reads as:

$$\begin{aligned} R_{L,N}^{(1)}(\beta,\omega) &:= \sum_{n=N+1}^{2N} (\delta\omega)^n \sum_{k=1}^N (-1)^k \sum_{i_j\in\{1,2\}} \chi_k^n(i_1,...,i_k) \\ &\cdot I_{k,L}(i_1,...,i_k)(\beta,\omega) + (-1)^{N+1} J_{N+1,L}(\beta,\omega). \end{aligned} \tag{2.14}$$

Proof. This proposition was proved in [4] in the sense of bounded operators. Again by using Remark 3.4 in [4], the operator $\widetilde{W}_L(\beta,\omega)$ as well as $W_{L,n}(\beta,\omega)$ belong to $B_1(L^2(\Lambda))$. The same argument holds for the remainder $R^{(1)}_{L,N}(\beta,\omega)$. The fact that the operator $\omega \to \frac{1}{(\delta\omega)^{N+1}} R^{(1)}_{L,N}(\beta,\omega)$ is a smooth B_1 operator valued function near ω_0 follows from definitions (2.14), (2.9) and (2.10). □

Remark 2.2. *It is important to note that the coefficients $W_{L,N}$ in (2.13) still depend on ω, but only through the magnetic phases. Let $\omega \in \mathbb{R}$. From the estimates* (2.7) *and the definitions in* (2.9) *and* (2.10), *and after integration over the τ variables, clearly we get as bounded operators:*

$$||R^{(1)}_{L,N}(\beta,\omega)|| \le \text{const } |\delta\omega|^{N+1}, \tag{2.15}$$

where the constant is uniform in $L > 1$.

3 Regularized expansion of g_L

We will now try to get a similar expansion for g_L, aiming to obtain (1.14). Fix $\beta > 0$ and $\omega_0 > 0$. The parameters z and ξ which enter the definition of g_L are fixed as in (1.5), and the estimates we make must be uniform w.r.t. them. Here ω is real and $\delta\omega$ is as above. It was shown in [5] that:

Lemma 3.1. *Let $\omega \in \mathbb{R}$. The trace class operator g_L admits a continuous integral kernel. Moreover, there exist two positive constants C and α, both independent of L, such that the integral kernel satisfies*

$$|g_L(\mathbf{x},\mathbf{x}';\omega)| \le C\, e^{-\alpha|\mathbf{x}-\mathbf{x}'|}. \tag{3.1}$$

Looking at (1.5), we see that we need a regularized expansion for the operator $(\xi - zW_L(\beta,\omega))^{-1}$. Using (2.15) with $N = 1$ we get for $\omega \in \mathbb{R}$,

$$\|W_L(\beta,\omega) - \widetilde{W}_L(\beta,\omega)\| \le C_1\, |\delta\omega|, \tag{3.2}$$

where C_1 is a L independent constant . Hence choose $|\delta\omega|$ is small enough such that

$$C_1 M|\delta\omega| \;\; (\sup_{z\in K} |z|) \;\; < 1/2, \tag{3.3}$$

C_1 and M being respectively defined in (3.3) and (1.3). Then the operator $1 - z(\widetilde{W}_L - W_L)(\xi - zW_L)^{-1}$ is invertible and its inverse has a norm less than 2. Hence by choosing Ω to be a small enough interval around ω_0 we get

$$\sup_{L>1}\sup_{\omega\in\Omega}\sup_{\xi\in\mathcal{C}_K}\sup_{z\in K}\left\|[\xi - z\widetilde{W}_L(\beta,\omega)]^{-1}\right\| \le 2M. \tag{3.4}$$

Then we can write:

$$\begin{aligned}
&(\xi - zW_L(\beta,\omega))^{-1} \\
&= (\xi - z\widetilde{W}_L(\beta,\omega))^{-1} \sum_{n=0}^{\infty} z^n \{[W_L(\beta,\omega) - \widetilde{W}_L(\beta,\omega)](\xi - z\widetilde{W}_L(\beta,\omega))^{-1}\}^n,
\end{aligned} \tag{3.5}$$

and thus we reduced the problem to the study of $(\xi - z\widetilde{W}_L(\beta,\omega))^{-1}$.

In order to get a convenient expansion for this inverse, we need to introduce some new notation. Let $\mathbf{x},\mathbf{x}' \in \Lambda$. Define for $N \ge 1$:

$$r_{L,N}(\mathbf{x},\mathbf{x}';\beta) := -z\int_\Lambda \frac{[i\,\mathrm{fl}(\mathbf{x},\mathbf{y},\mathbf{x}')]^N}{N!}\, G_L(\mathbf{x},\mathbf{y};\omega_0) g_L(\mathbf{y},\mathbf{x}';\omega_0) d\mathbf{y}, \tag{3.6}$$

where

$$\text{fl}(\mathbf{x},\mathbf{y},\mathbf{x}') := \phi(\mathbf{x},\mathbf{y}) + \phi(\mathbf{y},\mathbf{x}') + \phi(\mathbf{x}',\mathbf{x}) = \frac{1}{2}\mathbf{e}_3 \cdot [(\mathbf{y}-\mathbf{x}) \wedge (\mathbf{y}-\mathbf{x}')]; \tag{3.7}$$

with $\mathbf{e}_3 = (0,0,1)$, denotes the magnetic flux through the triangle defined by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}'$. Similarly let $\omega \in \mathbb{R}$ and define the bounded operator $r_L(\beta,\omega)$ and $\hat{r}_L(\beta,\omega)$ whose kernel is given by:

$$\begin{aligned} r_L(\mathbf{x},\mathbf{x}';\beta,\omega) &= -z\int_\Lambda \left(e^{i\delta\omega \text{fl}(\mathbf{x},\mathbf{y},\mathbf{x}')} - 1\right) G_L(\mathbf{x},\mathbf{y};\beta,\omega_0) g_L(\mathbf{y},\mathbf{x}';\beta,\omega_0) d\mathbf{y}, \\ \hat{r}_L(\mathbf{x},\mathbf{x}';\beta,\omega) &= e^{i\delta\omega\phi(\mathbf{x},\mathbf{x}')} r_L(\mathbf{x},\mathbf{x}';\beta,\omega) \end{aligned} \tag{3.8}$$

Notice that $\hat{r}_L$ does not coincide with the regularization of r_L given by (2.2). The operator r_L and $\hat{r}_L$ are related to the operators $r_{L,N}$ and $\tilde{r}_{L,N}$ respectively by

$$r_L(\beta,\omega) = \sum_{k=1}^\infty (\delta\omega)^k r_{L,k}(\beta); \quad \hat{r}_L(\beta,\omega) = \sum_{k=1}^\infty (\delta\omega)^k \tilde{r}_{L,k}(\beta). \tag{3.9}$$

Note that by using the Schur Holmgren criterion, the Lemma 3.1, (2.5) and the fact that $\text{fl}(\mathbf{x},\mathbf{y},\mathbf{x}')$ is bounded from above by L^2 on Λ, we have the estimate:

$$\left\| r_L - \sum_{k=1}^N (\delta\omega)^k r_{L,k} = \sum_{k=N+1}^\infty (\delta\omega)^k r_{L,k} \right\| \leq \text{const } e^{L^2} |\delta\omega|^{N+1}, \tag{3.10}$$

for some numerical positive constant. The same estimate again holds true for $\hat{r}_L(\beta,\omega)$ and the corresponding series given in (3.9).

Let us now give some more precise estimates on the norms of these operators.

Proposition 3.2. *Fix $N \geq 1$. There exist a positive constants C_2 independent of $L > 1$ such that for all $\omega \in \mathbb{R}$,*

$$\max\left\{ \max_{k=1}^N ||r_{L,k}(\beta)||,\ |\delta\omega|^{-1}||r_L(\beta,\omega)|| \right\} \leq C_2, \tag{3.11}$$

and

$$\max\left\{ \max_{k=1}^N ||r_{L,k}(\beta)||_{B_2},\ |\delta\omega|^{-1}||r_L(\beta,\omega)||_{B_2} \right\} = C_2 \cdot L^{3/2}. \tag{3.12}$$

These estimates also hold true for the regularized operators in the sense of (2.2) *and* $\hat{r}_L(\beta,\omega)$.

Proof. First, note that $|\text{fl}(\mathbf{x},\mathbf{y},\mathbf{x}')| \leq |\mathbf{x}-\mathbf{y}|\,|\mathbf{y}-\mathbf{x}'|$, see (3.7). The kernels present in the $\mathbf{y}$ integral are localized near their diagonal, see (2.5) and (3.1). By extending the integral with respect to $\mathbf{y}$ over the whole $\mathbb{R}^3$, then using a fraction of the exponential decay in order to bound the polynomial growth from the flux, we obtain a constant independent of L such that

$$|r_{L,k}(\mathbf{x},\mathbf{x}';\beta)| \leq C_3 \cdot e^{-\frac{\alpha}{4}|\mathbf{x}-\mathbf{x}'|}; \quad (\mathbf{x},\mathbf{x}') \in \Lambda^2, \quad 1 \leq k \leq N. \tag{3.13}$$

The same estimate holds for r_L. Now we can apply the Schur-Holmgren criterion (2.3) and get (3.11). The Hilbert-Schmidt estimates is also straightforward.

□

The next proposition gives the necessary expansion of $(\xi - z\widetilde{W_L}(\beta,\omega))^{-1}$.

Proposition 3.3. *Fix $N \geq 1$ and $\omega_0 > 0$. Then if $|\delta\omega|$ is small enough, the following identity holds in $B(L^2(\Lambda))$:*

$$\begin{aligned} (\xi - z\widetilde{W}_L(\beta,\omega))^{-1} &= \xi^{-1}\,(1 + \widetilde{g}_L(\beta,\omega))\ [1 + \xi^{-1}\hat{r}_L(\beta,\omega)]^{-1} \\ &= \xi^{-1}\,(1 + \widetilde{g}_L(\beta,\omega)) + \sum_{n=1}^N (\delta\omega)^n\, S_{L,n}(\beta,\omega) + R^{(2)}_{L,N}(\beta,\omega), \end{aligned} \tag{3.14}$$

where $S_{L,N}$ is given by

$$S_{L,N}(\beta,\omega) := \xi^{-1}\sum_{n=1}^{N}(-\xi^{-1})^n \sum_{(i_1,...,i_n)\in(N^*)^n} \chi_n^N(i_1,...,i_n)\,(1+\widetilde{g}_L(\beta,\omega))$$
$$\cdot\,\widetilde{r}_{L,i_1}(\beta,\omega)...\widetilde{r}_{L,i_n}(\beta,\omega) \tag{3.15}$$

where the remainder $R^{(2)}_{L,N}(\beta,\omega)$ has the property that the bounded operator valued function $\omega\to(\delta\omega)^{-N}R^{(2)}_{L,N}(\beta,\omega)$ is smooth around ω_0 and moreover, there exists a constant (possibly) depending on L such that

$$\|R^{(2)}_{L,N}(\beta,\omega)\| \leq \text{ const } |\delta\omega|^{N+1}. \tag{3.16}$$

Proof. We start with the following resolvent equation,

$$(\xi - z\widetilde{W}_L(\beta,\omega))^{-1} = \xi^{-1} + (\xi - z\widetilde{W}_L(\beta,\omega))^{-1}z\widetilde{W}_L(\beta,\omega)\xi^{-1}. \tag{3.17}$$

Now the next identity is very important, and it is obtained by a straightforward calculation from (3.7) and the definition of g_L (see also Proposition 13 in [5])

$$[\xi - z\widetilde{W}_L(\beta,\omega)]\widetilde{g}_L(\beta,\omega) = z\widetilde{W}_L(\beta,\omega) + \hat{r}_L(\beta,\omega). \tag{3.18}$$

If one multiplies with an inverse both sides of the above equality we get:

$$(\xi - z\widetilde{W}_L(\beta,\omega))^{-1}z\widetilde{W}_L(\beta,\omega) = \widetilde{g}_L(\beta,\omega) - (\xi - z\widetilde{W}_L(\beta,\omega))^{-1}\hat{r}_L(\beta,\omega). \tag{3.19}$$

We know from Proposition 3.2 that we can find a constant C_2 independent of L such that

$$\|\hat{r}_L(\beta,\omega)\| \leq C_2\,|\delta\omega|. \tag{3.20}$$

Let us use (3.19) in (3.17), and isolate the inverse we are interested in

$$[\xi - z\widetilde{W}_L(\beta,\omega)]^{-1}[1+\xi^{-1}\hat{r}_L(\beta,\omega)] = \xi^{-1}[1+\widetilde{g}_L(\beta,\omega)]. \tag{3.21}$$

Now if $|\delta\omega|$ is small enough, $1+\xi^{-1}\hat{r}_L(\omega)$ is invertible and (3.14) follows. Moreover, expressing the inverse by a finite Neumann-type expansion,

$$[1+\xi^{-1}\hat{r}_L]^{-1} = \sum_{k=0}^{N}(-\xi)^{-k}\hat{r}_L^k + [1+\xi^{-1}\hat{r}_L]^{-1}(-\xi)^{-(N+1)}\hat{r}_L^{(N+1)},$$

and using (3.9) we can identify the operators $S_{L,N}(\beta,\omega)$ as given in (3.15), while the reminder reads as

$$R^{(2)}_{L,N}(\beta,\omega) := (-\xi)^{-(N+1)}[\xi - z\widetilde{W}_L(\beta,\omega)]^{-1}\hat{r}_L^{N+1}(\beta,\omega) + \xi^{-1}[1+\widetilde{g}_L(\beta,\omega)]$$
$$\cdot\sum_{k=N+1}^{\infty}(\delta\omega)^k\sum_{n=1}^{N}(-\xi^{-1})^n\sum_{(i_1,...,i_n)\in(N^*)^n}\chi_n^k(i_1,...,i_n)\widetilde{r}_{L,i_1}(\beta,\omega)...\widetilde{r}_{L,i_n}(\beta,\omega). \tag{3.22}$$

Let us now identify the term in $(\delta\omega)^{N+1}$ which appears in the estimate (3.16). For the first term of the remainder it comes from (3.20), while for the second one it comes from the fact that the series begin with the index $N+1$ (see (3.10)). □

We are now ready to give a convenient expansion for the operator $[\xi - zW_L(\beta,\omega)]^{-1}$. First we need some new notation. We introduce the following operators,

$$S_{L,0}(\beta,\omega) := \xi^{-1}[1+\widetilde{g}_L(\beta,\omega)], \tag{3.23}$$

$$T_{L,N}(\beta,\omega) := \sum_{n=1}^{N}z^n\sum_{0\leq i_k\leq N,1\leq j_k\leq N}\chi_{2n+1}^N(i_0,j_1,i_1,...,j_n,i_n)S_{L,i_0}(\beta,\omega)$$
$$\cdot\,W_{L,j_1}(\beta,\omega)S_{L,i_1}(\beta,\omega)\ldots W_{L,j_n}(\beta,\omega)S_{L,i_n}(\beta,\omega),\quad N\geq 1. \tag{3.24}$$

Since the operators $W_{L,j}$ and $S_{L,i}$ defined in Propositions 2.1 and 3.3 are uniformly bounded in L, this is also true for $T_{L,N}$.

Corollary 3.4. *Fix $N \geq 1$ and $\omega_0 \geq 0$. If $|\delta\omega|$ is small enough, then the following identity holds in $B(L^2(\Lambda))$:*

$$[\xi - zW_L(\beta,\omega)]^{-1} = [\xi - z\widetilde{W}_L(\beta,\omega)]^{-1} + \sum_{n=1}^{N} (\delta\omega)^n \, T_{L,n}(\beta,\omega) + R^{(3)}_{L,N}(\beta,\omega), \tag{3.25}$$

where the remainder $R^{(3)}_{L,N}(\beta,\omega)$ has the property that the bounded operator valued function $\omega \to \frac{1}{(\delta\omega)^N} R^{(3)}_{L,N}(\beta,\omega)$ is smooth near ω_0, and there exists a constant (possibly) depending on L such that:

$$\|R^{(3)}_{L,N}(\beta,\omega)\| \leq \text{ const} \, |\delta\omega|^{N+1}. \tag{3.26}$$

Proof. The result follows after inserting the estimates from the previous proposition into formula (3.5), having used the notation introduced in (3.23), (3.24), (2.12) and (2.13). The rest is just a tedious bookkeeping of various terms. □

We finally are in the position of writing "the right" expansion for the operator $g_L(\beta,\omega)$ as announced in (1.14).

Theorem 3.5. *Fix $N \geq 1$ and $\omega_0 \geq 0$. If $|\delta\omega|$ is small enough, then the following equality takes place in $B_1(L^2(\Lambda))$:*

$$g_L(\beta,\omega) = g_{L,0}(\beta,\omega) + \sum_{n=1}^{N} (\delta\omega)^n \, g_{L,n}(\beta,\omega) + R^{(4)}_{L,N}(\beta,\omega), \tag{3.27}$$

where

$$g_{L,0}(\beta,\omega) := [\xi - z\widetilde{W}_L(\beta,\omega)]^{-1} z\widetilde{W}_L(\beta,\omega). \tag{3.28}$$

and $g_{L,n}$ are given by ($N \geq 1$),

$$g_{L,N}(\beta,\omega) := \sum_{n=1}^{N} \left[S_{L,N-n}(\beta,\omega) zW_{L,n}(\beta,\omega) \; + T_{L,n}(\beta,\omega) zW_{L,N-n}(\beta,\omega) \right], \tag{3.29}$$

where $W_{L,0} := \widetilde{W}_L$ and the remainder $\frac{1}{(\delta\omega)^N} R^{(4)}_{L,N}(\beta,\omega)$ has the property that the B_1 operator valued function $\omega \to \frac{1}{(\delta\omega)^N} R^{(4)}_{L,N}(\beta,\omega)$ is smooth near ω_0 and there exists a positive constant (possibly) depending on L such that:

$$\|R^{(4)}_{L,N}(\beta,\omega)\|_{B_1} \leq \text{ const} \, |\delta\omega|^{N+1} L^3. \tag{3.30}$$

Proof. First we multiply the $B_1(L^2(\Lambda))$ expansion (2.12) of the semigroup with the expansion (3.25) of the resolvent valid in $B(L^2(\Lambda))$. Thus one obtains in $B_1(L^2(\Lambda))$,

$$\begin{aligned}
g_L(\beta,\omega) &= [\xi - z\widetilde{W}_L(\beta,\omega)]^{-1} z \left(\widetilde{W}_L(\beta,\omega) + \sum_{n=1}^{N} (\delta\omega)^n W_{L,n}(\beta,\omega) \right) \\
&+ \sum_{n=1}^{N} (\delta\omega)^n \, T_{L,n}(\beta,\omega) z\widetilde{W}_L(\beta,\omega) + \sum_{n=1}^{N} \sum_{k=1}^{N} (\delta\omega)^{n+k} \, T_{L,n}(\beta,\omega) zW_{L,k}(\beta,\omega) \\
&+ [\xi - zW_L(\beta,\omega)]^{-1} z R^{(1)}_{L,N}(\beta,\omega) + R^{(3)}_{L,N}(\beta,\omega) z \left(\sum_{n=1}^{N} (\delta\omega)^n \, W_{L,n}(\beta,\omega) \right) \\
&+ R^{(3)}_{L,N}(\beta,\omega) z\widetilde{W}_L(\beta,\omega).
\end{aligned} \tag{3.31}$$

The last two lines will give a remainder $R^{(4_a)}_{L,N}(\beta,\omega)$, whose properties can be read out of those of the previous ones. This remainder has the same properties as $R^{(4)}_{L,N}(\beta,\omega)$, and in fact it is a part of it. The rest of the proof is just algebra, and amounts to identify the right factors which enter the definition of $g_{L,n}(\beta,\omega)$ and the expression of the full remainder. Here one must use (3.14) and the notation introduced in (3.15), (3.23) and (3.24). The proof is over. □

4 Expansion of the trace of g_L. The uniform bound.

We have almost all ingredients needed for proving (1.15). Let $\beta > 0$, $\omega_0 > 0$ and $\omega \in \Omega$ as in the Section 1.1. We now need to take the trace in (3.27). Let us begin with the trace of the operator $g_{L,0}(\beta,\omega)$ (see (3.28)). If we use (3.19), and then (3.14) , (3.17) and (3.23), we can write:

$$
\begin{aligned}
&g_{L,0}(\beta,\omega) = \widetilde{g}_L(\beta,\omega) - \xi^{-1}\hat{r}_L(\beta,\omega) - \xi^{-1}[\xi - z\widetilde{W}_L(\beta,\omega)]^{-1} z\widetilde{W}_L(\beta,\omega)\hat{r}_L(\beta,\omega) \\
&= \{\widetilde{g}_L(\beta,\omega) - \xi^{-1}\hat{r}_L(\beta,\omega)\} - \xi^{-1} z \sum_{k=0}^{N} (\delta\omega)^k S_{L,k}(\beta,\omega)\widetilde{W}_L(\beta,\omega)\hat{r}_L(\beta,\omega) \\
&- \xi^{-1} z R^{(2)}_{L,N}(\beta,\omega)\widetilde{W}_L(\beta,\omega)\hat{r}_L(\beta,\omega). \qquad (4.1)
\end{aligned}
$$

Apriori, this identity only holds in the bounded operators sense. But we know that $\widetilde{W}_L(\beta,\omega)$ is a trace class operator. It means that the operator

$$
M(\beta,\omega) := \widetilde{g}_L(\beta,\omega) - \xi^{-1}\hat{r}_L(\beta,\omega) \qquad (4.2)
$$

is a trace class operator, since all other operators in (4.1) are trace class. Note that the two individual terms in $M(\beta,\omega)$ might not be trace class. Now since $M(\beta,\omega)$ has a continuous integral kernel $M(\cdot,\cdot;\beta,\omega)$ (see Lemma 3.1 and (3.8)), its trace will be given by:

$$
\begin{aligned}
\mathrm{Tr}\ M(\beta,\omega) &= \int_\Lambda M(\mathbf{x},\mathbf{x};\beta,\omega)d\mathbf{x} \qquad (4.3)\\
&= \int_\Lambda \widetilde{g}_L(\mathbf{x},\mathbf{x};\beta,\omega)d\mathbf{x} - \xi^{-1}\int_\Lambda \hat{r}_L(\mathbf{x},\mathbf{x};\beta;\omega)d\mathbf{x} \\
&= \int_\Lambda g_L(\mathbf{x},\mathbf{x};\beta,\omega_0)d\mathbf{x} - \xi^{-1}\int_\Lambda r_L(\mathbf{x},\mathbf{x};\beta,\omega)d\mathbf{x}.
\end{aligned}
$$

The last line is very important, since it shows that the "tilde" disappears when we take the trace. This is because the magnetic phase $\phi(\mathbf{x},\mathbf{x}) = 0$ for all $\mathbf{x}$. But now $\int_\Lambda g_L(\mathbf{x},\mathbf{x};\beta,\omega_0)d\mathbf{x} =$ Tr $g_L(\beta,\omega_0)$, and we here recognize the very first term on the right hand side of (1.13). Now if we use (3.6) (3.8) and (3.9) we can write:

$$
\begin{aligned}
\mathrm{Tr}\ M(\beta,\omega) &= \mathrm{Tr}\ g_L(\beta,\omega_0) - \xi^{-1}\sum_{n=1}^{N}(\delta\omega)^n \int_\Lambda r_{L,n}(\mathbf{x},\mathbf{x};\beta,\omega_0)d\mathbf{x} \\
&+ (\delta\omega)^{N+1}\mathcal{R}^{(1)}_L(\beta,\omega,N), \qquad (4.4)
\end{aligned}
$$

where $\omega \to \mathcal{R}^{(1)}_L(\beta,\omega,N)$ is a smooth function in ω near ω_0. Moreover, due to (3.13) we obtain that the above integrals grow at most like L^3, as required.

But there are several other terms which remain to be considered in (4.1) and (3.27). They are respectively Tr $\{S_{L,k}(\beta,\omega)\widetilde{W}_L(\beta,\omega)\widetilde{r}_L(\omega)\}$ and Tr $g_{L,n}(\beta,\omega)$.

These traces have two important things in common. First, we always take the trace of a product of integral operators with continuous kernels. Second, they all still depend on $\delta\omega$, but only through the magnetic phases; all factors are regularized operators, as defined in (2.2). We will now try to discuss all these different terms in a unified manner.

Fix $\omega_0 > 0$. Let $\omega \in \mathbb{R}$ and $\delta\omega$ as above. Consider a product of operators of the form

$$
T(\omega) := \widetilde{T}_0(\omega)\widetilde{T}_1(\omega)...\widetilde{T}_n(\omega)
$$

where $\widetilde{T}_i(\omega)$ are the regularized operators associated to some integral operators $T_i(\omega_0)$, $i = 0,...,n$ (see (2.2)) and assume that this product is of trace class. Denote by $t_i(\cdot,\cdot)$ the kernel of $T_i(\omega_0)$, which is supposed to be jointly continuous in $\mathbf{x}$ and $\mathbf{x}'$. We denote by fl_n the following flux related

quantity

$$\mathrm{fl}_1(\mathbf{x},\mathbf{y}_1)=0,\quad \mathrm{fl}_n(\mathbf{x},\mathbf{y}_1,...,\mathbf{y}_n)=\sum_{k=1}^{n-1}\mathrm{fl}(\mathbf{x},\mathbf{y}_k,\mathbf{y}_{k+1})$$
$$=\phi(\mathbf{x},\mathbf{y}_1)+\phi(\mathbf{y}_1,\mathbf{y}_2)+...+\phi(\mathbf{y}_{n-1},\mathbf{y}_n)+\phi(\mathbf{y}_n,\mathbf{x}),\ n\geq 2. \tag{4.5}$$

Another important property of these operators is that their kernels are exponentially localized near the diagonal (see (2.5), (3.1) and (3.13)). Therefore there exist two positive constants C and α, independent of L, such that:

$$\max_{i=0}^{n}|t_i(\mathbf{x},\mathbf{x}')|\leq C\,e^{-\alpha|\mathbf{x}-\mathbf{x}'|},\quad (\mathbf{x},\mathbf{x}')\in\Lambda^2. \tag{4.6}$$

Then the diagonal value of the kernel of $T(\omega)$ reads as

$$T(\mathbf{x},\mathbf{x},\omega)=\int_\Lambda d\mathbf{y}_1...\int_\Lambda d\mathbf{y}_n\, e^{i\delta\omega \mathrm{fl}_n(\mathbf{x},\mathbf{y}_1,...,\mathbf{y}_n)}t_0(\mathbf{x},\mathbf{y}_1,\omega_0)t_1(\mathbf{y}_1,\mathbf{y}_2,\omega_0)...$$
$$t_{n-1}(\mathbf{y}_{n-1},\mathbf{y}_n,\omega_0)t_n(\mathbf{y}_n,\mathbf{x},\omega_0), \tag{4.7}$$

where we added together all individual phases from each regularized factor. Because we assumed that T is a trace class operator, the trace of T is

$$\mathrm{Tr}\,T(\omega)=\int_\Lambda T(\mathbf{x},\mathbf{x},\omega)d\mathbf{x}. \tag{4.8}$$

For $m\geq 0$, $n\geq 1$, let us introduce the notation:

$$d_{m,n}(L):=\int_\Lambda d\mathbf{x}\int_\Lambda d\mathbf{y}_1...\int_\Lambda d\mathbf{y}_n[i\mathrm{fl}_n(\mathbf{x},\mathbf{y}_1,...,\mathbf{y}_n)]^m\,t_0(\mathbf{x},\mathbf{y}_1,\omega_0)$$
$$t_1(\mathbf{y}_1,\mathbf{y}_2,\omega_0)...t_{n-1}(\mathbf{y}_{n-1},\mathbf{y}_n,\omega_0)t_n(\mathbf{y}_n,\mathbf{x},\omega_0). \tag{4.9}$$

Lemma 4.1. *For every $m\geq 0$ and $n\geq 1$, there exists a constant independent of L but depending on m,n such that*

$$|d_{m,n}(L)|\leq \mathrm{const}L^3\,. \tag{4.10}$$

Moreover, for a given $N\geq 1$ we have

$$\mathrm{Tr}\;T(\omega)=\sum_{m=0}^{N}(\delta\omega)^m d_{m,n}(L)+(\delta\omega)^{N+1}\mathcal{R}_L(\omega,N), \tag{4.11}$$

where $\omega\to\mathcal{R}_L(\omega,N)$ is a smooth function near ω_0.

Proof. The equality (4.11) comes straight out of (4.7) and (4.8).

Now let us prove the estimate (4.10). We recall the following estimate (3.7) on the magnetic flux,

$$|\mathrm{fl}(\mathbf{x},\mathbf{y},\mathbf{z})|\leq|\mathbf{x}-\mathbf{y}||\mathbf{y}-\mathbf{z}|. \tag{4.12}$$

Then by induction one has for all $n\geq 1$,

$$|\mathrm{fl}_n(\mathbf{x},\mathbf{y}_1,...,\mathbf{y}_n)|\leq\left(|\mathbf{x}-\mathbf{y}_1|+|\mathbf{y}_1-\mathbf{y}_2|+...+|\mathbf{y}_{n-1}-\mathbf{y}_n|\right)^2. \tag{4.13}$$

Therefore the polynomial growth induced by this flux is diagonalized, i.e. it only depends on differences between the variables y_i,y_{i+1}. But due to (4.6), we can write

$$|d_{m,n}(L)|\leq\mathrm{const}\int_\Lambda d\mathbf{x}\int_\Lambda d\mathbf{y}_1...\int_\Lambda d\mathbf{y}_n e^{-\frac{\alpha}{2}|\mathbf{x}-\mathbf{y}_1|}$$
$$e^{-\frac{\alpha}{2}|\mathbf{y}_1-\mathbf{y}_2|}...e^{-\frac{\alpha}{2}|\mathbf{y}_{n-1}-\mathbf{y}_n|}e^{-\frac{\alpha}{2}|\mathbf{y}_n-\mathbf{x}|}, \tag{4.14}$$

for some L independent constant. Here we used the exponential decay to bound the polynomial factors and then we extend the $\mathbf{y}$ integrals to the whole $\mathbb{R}^3$. So the volume growth is only given by the integral over $\mathbf{x}$ in the r.h.s of (4.14). The proof the lemma is over. □

4.1 Proof of (1.13) and of Theorem 1.1

We can now put together the results of this section and prove the key estimate (1.13). In Theorem 3.5 we obtained an expansion for $g_L(\beta,\omega)$ as announced in (1.14). When we take the trace of $g_L(\beta,\omega)$, the term $R^{(4)}_{L,N}(\beta,\omega)$ will only give a contribution to the remainder in (1.13), hence we ignore it.

Then the term $g_{L,0}(\beta,\omega)$ given in (3.28) can be written as a sum between an operator $M(\beta,\omega)$ from (4.2), and a sum of operators of the type treated in Lemma 4.1. Then from (4.4) and the above mentioned lemma we can conclude that $(\partial_\omega^N \mathrm{Tr}\ g_{L,0})(\beta,\omega_0)$ grows at most like L^3.

Finally, looking at the contribution coming from $g_{L,n}(\beta,\omega)$, with $n \geq 1$. Using the same lemma, we obtain in a similar way that $(\partial_\omega^N \mathrm{Tr}\ g_{L,n})(\beta,\omega_0)$ grows at most like the volume. We therefore conclude that $(\partial_\omega^N \mathrm{Tr}\ g_L)(\beta,\omega_0)$ behaves like L^3, uniformly in ξ and z, and the proof of (1.12) is done.

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